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A basis for comparing the sensitivities of different electromagnetic flowmeters to velocity distribution

By J. HEMP† AND D. G. WYATT

The Nuffield Institute for Medical Research, University of Oxford,
Headley Way, Headington, Oxford OX3 9DS

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The response of an electromagnetic flowmeter to the flow rate through it should ideally be independent of the velocity distribution of the liquid within it. In general such flowmeters do not exhibit this ideal behaviour and the variation in response with different velocity distributions, for a given flow rate, can be very large. The sensitivity of a flowmeter to velocity distribution is here described in terms of a 'worst flow', which is a particular velocity distribution peculiar to a given flowmeter. The 'worst flow' is described mathematically in terms of functions which depend upon the design parameters of the flowmeter. The flowmeter response to its 'worst flow', for a given kinetic energy of motion, is formulated. This enables a criterion (ϵ) to be defined which describes the sensitivity of a given flowmeter to velocity profile effects and which permits different flowmeters to be compared in this respect. Methods of evaluating ϵ are discussed and its value is calculated for a conventional flowmeter, which employs small electrodes and an approximately uniform magnetic field. It is shown that it is possible actually to generate this worst flow in a given flowmeter, by operating the flowmeter in the role of an electromagnetic pump. This could lead to the direct measurement of ϵ. The possibility is discussed of other definitions of ϵ, based on different boundary conditions and constraints and which might lead to a less severe criterion.

1. Introduction

Principle of the electromagnetic flowmeter. In its conventional form the electromagnetic flowmeter consists of a short cylindrical flow channel of circular cross-section, in the wall of which are fitted two small diametrically opposed electrodes, their surfaces in contact with the flowing liquid. A suitable external magnetic field is imposed upon the channel in the general region occupied by the electrodes, between which an electric potential is produced by the movement of the conducting liquid through the magnetic field. This electric potential is amplified and recorded as a measure of flow rate through the channel.

The conventional electromagnetic flowmeter has a high degree of sensitivity, for a given flow rate, to the distribution of velocity within the liquid (see, for example, Shercliff 1962; Wyatt 1972, 1977; Al-Khazraji *et al.* 1978). Hitherto there has been no

† Present address: Fluid Engineering Unit, Cranfield Institute of Technology, Cranfield, Bedford MK43 0AL.

description of a rigorous analytical basis for comparing the sensitivities of different flowmeters to three-dimensional velocity distribution effects, although the suggestion has been mentioned previously (Wyatt 1972, 1977). It is the purpose of this paper to describe such a basis and to indicate how it can be evaluated. We note that an approximate and different numerical basis has been used elsewhere (Al-Khazraji 1979).

The basic flowmeter equation (Shercliff 1962) is

$$\nabla^2 U = \nabla . (\mathbf{v} \times \mathbf{B}) \tag{1}$$

where U is the electric potential in the liquid, $\mathbf{v}$ is the velocity and $\mathbf{B}$ the magnetic flux density. The most general solution to this equation is that given by Bevir (1970):

$$U = \int_\tau \mathbf{W} . \mathbf{v} \, d\tau, \tag{2}$$

where

$$\mathbf{W} = \mathbf{B} \times \mathbf{j}. \tag{3}$$

Here $\mathbf{j}$ is a particular current distribution which describes completely the boundary conditions of the flow channel and electrodes of a given flowmeter. Precisely, it is the current density that would be set up in the liquid by passing unit current into one electrode and extracting it from the other. Bevir gave the name *virtual current* to $\mathbf{j}$, to distinguish it from currents which exist in a working flowmeter. $\mathbf{W}$ is known as the *weight vector*.

The volume τ over which the integral is taken is theoretically the whole liquid volume but in practice can be taken to be the volume of liquid in the flowmeter together with the liquid in those parts of the connecting pipes on either side of it from beyond which contributions to the integral are negligible. Provided the magnetic induction falls with distance from the flowmeter, the distance either side of the flowmeter which contributes to τ is governed by the virtual current and is at most 3 pipe diameters for 0·1 % accuracy (Bevir 1970; Hemp 1975, figure 10).

In addition to equation (2), Bevir also gave the following condition for a flowmeter to be ideal, that is, for its response to flow rate to be entirely independent of the velocity distribution:

$$\nabla \times \mathbf{W} = 0. \tag{4}$$

Note that in this particular case, when the vector field represented by $\mathbf{W}$ is irrotational, we can write

$$\mathbf{W} = \nabla \phi, \tag{5}$$

in which ϕ is a scalar quantity.

Bevir's results (2), (3) and (4) form the basis of flowmeter theory and what follows here is dependent upon them.

2. The 'worst flow' concept

2.1. *Definition of 'worst flow'*

It is known that, in general, the response of an electromagnetic flowmeter to a given flow rate through it varies with the velocity distribution (or, as is often said, with the velocity profile; although it is to be noted that this term implies that the flow is rectilinear, which generally is not so). We refer to this response variation as 'error' and it is natural to enquire whether there is not a particular velocity distribution which causes greater 'error' than any other. (We would clearly need a baseline against which to measure such 'error': a natural one to choose would be the response with a uniform velocity.) Now any velocity distribution can be resolved into a uniform velocity due to the flow rate and a purely circulating velocity distribution having zero flow rate through the meter. Thus our enquiry reduces to the question of whether there is a particular circulating velocity distribution having zero flow rate through the meter which would elicit a larger response than any other such velocity distribution (subject of course to some appropriate constraint on its overall magnitude). This we tentatively term the 'worst flow'.

2.2. *Mathematical development of the 'worst flow'*

If such a 'worst flow' exists it should be possible to define it by finding that purely circulating velocity distribution which maximizes the magnitude of

$$U = \int_\tau \mathbf{W}.\mathbf{v}\,d\tau$$

subject to any given constraints on the velocity and under the appropriate boundary conditions. We consider a flowmeter in the middle of a straight pipe which extends to infinity in both directions. We restrict the kinetic energy of the motion by applying the relationship

$$\int_\tau v^2\,d\tau = K, \tag{6}$$

where K is a constant, this being the easiest way to constrain the velocities in an unknown velocity distribution (this and other possible ways of constraining the velocity are discussed in §7). The volume τ is bounded by part of the pipe wall of area S_w and by two arbitrary imaginary surfaces S_u and S_d which span the pipe upstream and downstream from the flowmeter respectively. In order to make the best use of the available kinetic energy the worst flow will clearly concentrate itself in the central region of the flowmeter where the sensitivity is high; but we cannot from this observation rule out the possibility of finite $\mathbf{v}$ at S_u and S_d. However, if we choose to place S_u and S_d infinitely far from the flowmeter we can say with certainty that $\mathbf{v} = 0$ at these surfaces. The reasons are: first, any movement at the surfaces which is associated with movement near the flowmeter necessarily implies movement throughout the pipe and therefore would be prevented by the imposition of equation (6). Secondly, any circulating movement restricted to the locality of the surfaces (which therefore would not be prevented by equation (6)) would not occur because all practical flowmeters have $\mathbf{W} = 0$ in regions infinitely far from the flowmeter. Consequently such movements would generate no signal and would not be invoked by the maximization process. In what follows, the surfaces S_u and S_d will be taken to be infinitely far from the flowmeter and the magnitude of $\mathbf{v}$ on these surfaces will be taken as zero. All volume integrals will be taken over the whole, infinite volume of the pipe and for convenience the bound τ will be dropped. We note that equation (6), together with the choice of S_u, S_d at infinity, also precludes the possibility of a finite flowrate and ensures that the resulting velocity distribution is a purely circulating one.

Our problem therefore is to maximize U with the boundary conditions

$$\mathbf{v}.d\mathbf{s} = 0 \quad \text{on the wall } S_w, \tag{7}$$

$$\mathbf{v} = 0 \quad \text{on the surfaces } S_u, S_d, \tag{8}$$

and under the constraints

$$\int v^2\,d\tau = K, \tag{9}$$

$$\nabla.\mathbf{v} = 0. \tag{10}$$

We have to find the extremum of the function

$$L = \int \mathbf{W}.\mathbf{v}\,d\tau + \int \lambda(\nabla.\mathbf{v})\,d\tau + \gamma\left[\int v^2\,d\tau - K\right],$$

where λ and γ are Lagrangian multipliers: λ is a function of position and γ is a constant. We use the identity

$$\int \lambda(\nabla.\mathbf{v})\,d\tau = \int \lambda\mathbf{v}.\,d\mathbf{s} - \int \mathbf{v}.(\nabla\lambda)\,d\tau \tag{11}$$

so that L may be written

$$L = \int (\mathbf{W}-\nabla\lambda).\mathbf{v}\,d\tau + \gamma\left[\int v^2\,d\tau - K\right]. \tag{12}$$

We set the variation of L equal to zero, whence

$$0 = \int (\mathbf{W}-\nabla\lambda+2\gamma\mathbf{v}).\,\delta\mathbf{v}\,d\tau \tag{13}$$

and thus

$$\mathbf{v} = -\frac{1}{2\gamma}(\mathbf{W}-\nabla\lambda). \tag{14}$$

We use equation (14) with equations (7), (9) and (10) to obtain the following equations which define λ and γ (Bevir's (1970) theory assumes that $\nabla\times\mathbf{B}$ and $\nabla\times\mathbf{j}$ are both zero; hence $\nabla.\mathbf{W} = \mathbf{j}.\nabla\times\mathbf{B}-\mathbf{B}.\nabla\times\mathbf{j} = 0$):

$$\left.\begin{aligned} &\nabla^2\lambda = 0, \\ &\frac{\partial\lambda}{\partial n} = W_n \quad \text{on the wall,} \end{aligned}\right\} \tag{15}$$

$$\gamma = \pm\frac{1}{2}\left(\frac{\int(\mathbf{W}-\nabla\lambda)^2\,d\tau}{K}\right)^{\frac{1}{2}}, \tag{16}$$

where $\partial/\partial n$ denotes the derivative in a direction along the normal to the wall and W_n is the component of $\mathbf{W}$ in the same direction.

The extremum of U is obtained by substituting equations (14) and (16) into the equation for U. However, we first note that since

$$\int (\nabla\lambda).\mathbf{v}\,d\tau = \int \nabla.(\lambda\mathbf{v})\,d\tau - \int \lambda(\nabla.\mathbf{v})\,d\tau = \int \lambda\mathbf{v}.\,d\mathbf{s} = 0,$$

U may be written

$$U = \int \mathbf{W}.\mathbf{v}\,d\tau = \int (\mathbf{W}-\nabla\lambda).\mathbf{v}\,d\tau. \tag{17}$$

Equations (14), (16) and (17) give the extremum U_m of U:

$$U_m = \left(K\int (\mathbf{W}-\nabla\lambda)^2\,d\tau\right)^{\frac{1}{2}}. \tag{18}$$

We show in appendix C that U_m represents the maximum value of U. That is, the velocity distribution given by equation (14), where λ is defined by equations (15), is the 'worst flow'.

Note that, except for a constant, equation (14) is entirely dependent upon the design characteristics of the flowmeter, as expressed by $\mathbf{W}$. Thus each flowmeter has a particular 'worst flow' distribution. Also, the response U_m to the 'worst flow' is, for a given kinetic energy, purely a function of the flowmeter design.

2.3. *Interpretation of* $\int(\mathbf{W}-\nabla\lambda)^2\,d\tau$

The integral $\int(\mathbf{W}-\nabla\lambda)^2\,d\tau$ has the form of the sum of the squares of the deviations of $\mathbf{W}$ from the quantity $\nabla\lambda$; and, since $\nabla\lambda$ is the gradient of a scalar quantity, the integral is a measure of the deviation of $\mathbf{W}$ from a certain irrotational vector field. Now it can be shown that, if we consider $\mathbf{W}$ as given and λ as variable, the value of λ which *minimizes* the integral is precisely that defined by equations (15). That is, the integral is a measure of the deviation of $\mathbf{W}$ from the *nearest* irrotational vector field.

3. The basis for comparison of different flowmeters

We define the following quantity ϵ as a measure of the sensitivity of any electromagnetic flowmeter to velocity distribution:

$$\epsilon = \left(a^3\int(\mathbf{W}-\nabla\lambda)^2\,d\tau\right)^{\frac{1}{2}}\Big/\int W_z\,d\tau, \tag{19}$$

where the integrals are taken over the whole volume of liquid. W_z is the component of $\mathbf{W}$ parallel to the flowmeter axis and determines the flowmeter response to a uniform velocity distribution. The square root and a^3 (a is the radius of the flowmeter) ensure that ϵ is non-dimensional; the factor a^3 ensures that ϵ has the same value for flowmeters of the same design but which are of different sizes. ϵ has the form of the square root of the square of the deviation of the weight vector from the nearest irrotational vector field summed over the whole volume of liquid, as a fraction of W_z summed over the whole of the volume of the liquid.

Apart from the observations that we have made in §§2.1 and 2.3 and the way that ϵ follows naturally from them, we can see directly that ϵ is a measure of the degree to which the sensitivity of the flowmeter varies with velocity distribution. Suppose there to be a velocity distribution consisting of a circulating flow of zero flow rate, which by itself would produce a flowmeter voltage δU, together with a uniform velocity V which by itself would produce a voltage U. Then

$$\frac{\delta U}{U} \leqslant \frac{U_m}{U},$$

where U_m is the 'worst flow' with the same kinetic energy as the given circulating flow. Now $\int W_z\,d\tau$ is simply the sensitivity S of the flowmeter, defined here as the voltage obtained with a uniform flow of unit speed. Hence

$$\epsilon = \frac{U_m}{S(K/a^3)^{\frac{1}{2}}},$$

whence

$$\frac{U_m}{U} = \epsilon\frac{K/a^3}{V}$$

and

$$\frac{\delta U}{U} \leqslant \epsilon\frac{(K/a^3)^{\frac{1}{2}}}{V}. \tag{20}$$

Now $(K/a^3)^{\frac{1}{2}}$ is an average of the speed of the circulating flow and equation (20) therefore shows that ϵ is a measure of the degree to which the flowmeter voltage is affected by the circulating flow, i.e. by the velocity distribution for a given flow rate through the meter.

4. The magnitude of ϵ

It is well known that when the magnetic field is uniform the weight function $\mathbf{W}$ tends to infinity as a point electrode is approached. The integral $\int|\mathbf{W}|\,d\tau$ however remains finite, which means of course that the sensitivity ($\int W_z\,d\tau$) is also finite. This is merely a consequence of the way $\mathbf{W}$ varies as the electrode is approached and there is nothing curious about it either mathematically or physically. However, the integral

$$I = \int (\mathbf{W} - \nabla\lambda)^2\,d\tau$$

does not remain finite. The reason is that I contains $\int W^2\,d\tau$ and $\mathbf{W} \propto R^{-2}$, which ensures that I diverges (see appendix A, equation (A 3); it happens to be the case that $\nabla\lambda$ also is $\propto R^{-2}$ (appendix A, equation (A 8)) but the 'constant' of proportionality is different so that I definitely diverges). In this respect I behaves very much as the variance of the rectilinear weight function (Wyatt 1972).

In general then ϵ will assume values ranging from 0 for an ideal flowmeter to ∞ for a uniform-field, point-electrode flowmeter (see also appendix B). Clearly, a 'good' flowmeter will be one for which $\epsilon \ll 1$ (see equation (20)). Most medical and industrial flowmeters in use today have magnetic fields which are approximately uniform and electrodes which although not points are nevertheless small. As an example of the value of ϵ to be expected with such flowmeters, we have calculated that, when the electrode radius is $\frac{1}{20}$ of the channel radius, ϵ is approximately 0·63 (see appendix A). This value, for a flowmeter which is known to be highly sensitive to velocity distribution, is the maximum value of ϵ likely to be met with in practice. It serves as an upper figure with which values of ϵ for other (improved) flowmeters may be compared.

5. Generation of the 'worst flow'

It is possible to generate the 'worst flow' in the flowmeter itself. We were led to the means of achieving this through several consecutive observations. First $\mathbf{B}\times\mathbf{j}$, in addition to being the weight vector, is also the body force on the liquid when a real current is made to flow into one electrode and out of the other. Second, it occurred to us that equation (4) implies that the body force can be exactly balanced by a pressure distribution p so that $\mathbf{W} - \nabla p = 0$. This led to the following idea for testing whether or not a given meter is ideal. The meter is operated in the 'pump mode', that is, it is placed in a pipe the ends of which are closed far from the flowmeter and a current is passed between the electrodes with the magnetic field present. Motion of the liquid (indicated for example by suspended particles) would indicate that the flowmeter was not ideal; no motion would mean it was ideal.

One of us (DGW) then conceived the idea of a 'worst flow' and recognized that the liquid motion generated in the pump mode when the meter is not ideal must in some sense actually be the 'worst flow'. This is because the body force in the pump mode causes the greatest circulation in precisely those loops around which flow causes the greatest flowmeter signal.

When the flowmeter is in the pump mode, the equation governing the motion is:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}.\nabla)\mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} - i\mathbf{W} \tag{21}$$

subject to the conditions

$$\left.\begin{aligned} \mathbf{v}.d\mathbf{s} &= 0, \quad \text{at the walls,} \\ \nabla.\mathbf{v} &= 0, \end{aligned}\right\} \tag{22}$$

where $\mathbf{v}$ is the velocity and ρ the density of the liquid, p the pressure, η the viscosity, i the current passed to produce the motion and $\mathbf{W} = \mathbf{B} \times \mathbf{j}$ as previously. These equations are sufficient only when certain effects are small. First, the magnetic field due to the current i must be much less than the applied field $\mathbf{B}$. The order-of-magnitude condition for this is

$$\frac{\mu_0 i}{Bl} \ll 1, \tag{23}$$

where l is a characteristic length. Secondly, the e.m.f. induced by the motion of the liquid may cause currents only small in comparison with the applied current i. That is

$$\frac{\sigma v B l^2}{i} \ll 1, \tag{24}$$

where σ is the electrical conductivity of the liquid. These and other effects, such as thermal convection due to the heating effect of the current, contact impedance and tertiary magnetic fields could in practice be made negligible.

Suppose then with these assumptions that $\mathbf{v} = 0$ for $t < 0$ and that, at time $t = 0$, $\mathbf{W}$ suddenly assumes a constant value, i.e. a steady current and a steady magnetic field are switched on. Now provided the convective term $(\mathbf{v}.\nabla)\mathbf{v}$ in equation (21) is small, i.e.

$$\frac{vt}{l} \ll 1, \tag{25}$$

and provided the boundary layer is undeveloped, i.e.

$$\left(\frac{\eta t}{\rho l^2}\right)^{\frac{1}{2}} \ll 1, \tag{26}$$

equation (21) becomes

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - i\mathbf{W}. \tag{27}$$

At time $t = 0$, a pressure distribution is instantly created which, in view of equations (25) and (26), remains constant in time. Thus equation (27) may be integrated to give

$$\mathbf{v} = -\frac{1}{\rho}(i\mathbf{W} + \nabla p)t. \tag{28}$$

Equations for p may now be obtained from equations (28) and (22). Since $\nabla.\mathbf{W} = 0$ (see §2.2) we obtain

$$\left.\begin{aligned} \nabla^2 p &= 0, \\ \frac{\partial p}{\partial n} &= -iW_n \quad \text{on the wall.} \end{aligned}\right\} \tag{29}$$

Equations (28) and (29) are identical with equations (14) and (15) for the 'worst flow', the ratios $-p/i$ and $\rho/2t$ appearing instead of λ and γ respectively. That is to say, the velocity distribution given by equation (28), with the flowmeter in the pump mode and under the conditions given by equations (22) to (26), is identical with the 'worst flow' distribution given by equation (14).

The kinetic energy of the motion under these conditions is

$$K = \int v^2 \, d\tau = \frac{i^2 t^2}{\rho^2} \int (\mathbf{W} - \nabla\lambda)^2 \, d\tau. \tag{30}$$

6. Methods of evaluating ϵ

There are several methods for calculating ϵ, all of which depend on evaluating **B**, **j** and $\nabla\lambda$ throughout the liquid. In practice the integrals in equation (19) need not be evaluated very far along the pipe axis. Although in some instances the magnetic field may extend many radii from the plane of symmetry normal to the axis, the virtual current falls off rapidly in the direction of the pipe axis beyond the edges of the electrodes (Bevir 1970; Hemp 1975, figure 10). Consequently evaluation of the integrals over a volume bounded by the surface of the pipe and by planes separated by a distance equal to the electrode length plus 3 radii each side would give ample accuracy for the evaluation of ϵ.

Alternatively, ϵ could in principle (also we believe in practice) be found by generating the worst flow and measuring U_m.

6.1. *Calculation from analytic formulae*

This method is practicable only when the boundary conditions on **B** and **j** are simple enough to permit both formulation of **W** and $\nabla\lambda$ and evaluation of the integrals in equation (19). An example of this approach is given by appendix A.

6.2. *By finite differences*

Provided the boundary conditions on **B** and **j** are clearly defined it is possible to evaluate **B** and **j** throughout the liquid using the method of finite differences. **W** and $\nabla\lambda$ could then be computed from the values thus obtained.

6.3. *By experimental determination of* **B** *and/or* **j**

B and **j** can both be determined experimentally by measurement of their normal components at the surface bounding the flowmeter (Bevir, O'Sullivan & Wyatt 1981). Experimental procedures of this kind could be combined with either of the methods given above to yield **W** and thus $\nabla\lambda$.

6.4. *By direct measurement of* U_m

Suppose the worst flow is generated as described in §5 and that at time t the current i is stopped and the flowmeter mode instantly assumed. Equations (18), (19) and (30) yield

$$\epsilon = \left(\frac{a^3 \rho U_m}{S^2 i t}\right)^{\frac{1}{2}}. \tag{31}$$

All the quantities on the right-hand side of this equation are measurable; hence ϵ can be found.

We have examined the design of an experiment of this kind. Although we think it is feasible, problems arise both from the small value of U_m (typically 0·5 μV) and the relatively large bandwidth of the amplifier which is necessary to enable the signal to be measured within the period t. The poor signal-to-noise ratio which results calls for a very high standard of design and execution. Signal averaging would need to be used, which implies long periods between consecutive observations in order to allow the liquid to come to rest. We have considered alternative schemes, based on measuring the induced voltage due to an oscillating 'worst flow' or to a steady-state 'worst flow' calling for a different definition of ϵ: most are feasible but difficult.

The sensitivity S may be measured conventionally by passing liquid through the flowmeter at high Reynolds number with appropriate entrance and exit lengths. Alternatively, it may be measured by observing the pressure difference Δp which is generated between the closed ends when the flowmeter is in the pump mode. It may be shown, by integrating the z-component of equation (28), that

$$S = (A/i)\,\Delta p,$$

where A is the cross-section of the pipe.

7. Discussion

Hitherto there has been no criterion by which the sensitivities of different electromagnetic flowmeters to velocity profile effects may be judged. The concept, indeed the physical reality, of a 'worst flow' has enabled us to suggest such a criterion. The criterion is not a mathematical abstraction but represents a real quality of electromagnetic flowmeters. It can be computed and possibly directly measured.

The 'worst flow' that we have described is a velocity distribution which is determined by specified boundary conditions and by a particular constraint on the circulation, namely that it reflects a specified kinetic energy. However, other 'worst flows' are conceivable, that depend on different boundary conditions and on different constraints on the circulation. For example, we could impose the no-slip condition at the wall, or assume a particular boundary-layer structure. Either of these assumptions would more severely restrict the range of possible velocity distributions explored by the extremization process than our very limited boundary conditions have done. Again, we could assume that the speed of the liquid was everywhere constant, rather than limit the total kinetic energy.

There is therefore a number of 'worst flows', each of which could lead to a different definition of ϵ. The question arises whether any of these are more practical or more useful than others. The 'worst flow' that we have used can, when strictly applied, over-weight the 'worst' regions in the flowmeter. This is clear, for example, in the case of small hemispherical electrodes and a uniform magnetic field (appendix A) because, as $r_e \to 0$, $\epsilon \to \infty$, whereas we know that the sensitivity to velocity profile effects which are met in practice does not get greater in small-electrode meters as electrode size is diminished. However, the problem of diverging values of ϵ (appendix B) and consequent over-weighting can be overcome by terminating the integrals in a sensible manner. For example, if practical flow profiles are such that velocity does not change significantly over distances $\leqslant \frac{1}{20}$ of the pipe diameter, a hemisphere of this same dimension can be taken as representative of any electrode of this size or smaller. The relative ease of mathematical calculations is the advantage of the definition of ϵ we have used. Finally we note that as it will almost certainly be possible to design ideal or very nearly ideal meters by optimization of electrode and magnet shapes the tendency of any measure ϵ to over-weight some regions becomes unimportant because, however defined, its value will be small.

Appendix A. Estimate of ϵ for a uniform-field flowmeter with small diametrically opposite hemispherical electrodes

Let the tube radius be a, the radius of the hemispherical electrodes r_e and the magnetic induction B. The direction of the field is perpendicular to the duct axis and to the line joining the electrode centres (figure 1a). The sensitivity S is $2aB$.

It is assumed (and this will be verified later) that the main contribution to the integral in the numerator of equation (19) comes from regions of the channel in the vicinity of the electrodes. The integral is therefore evaluated for the simplified case of a hemispherical electrode on an insulated plane wall looking out on an infinite half-space of liquid with a uniform field parallel to the wall. The answer is doubled (since there are two electrodes) and substituted into equation (19) to give an estimate of ϵ. We shall use the transformation

$$\int (\mathbf{W}-\nabla\lambda)^2 d\tau = \int W^2 d\tau - \int \lambda \mathbf{W}.d\mathbf{s}. \tag{A 1}$$

We take spherical co-ordinates as shown in figure 1(b). Let the magnetic field be in the positive x direction. The wall is then the surface $\theta = \frac{1}{2}\pi$ and the liquid occupies the region $R > r_e, 0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The virtual current is

$$\mathbf{j} = \frac{-1}{2\pi R^2}\mathbf{u}_R, \tag{A 2}$$

so the weight vector is

$$\mathbf{W} = \frac{B}{2\pi R^2}(\sin\phi\,\mathbf{u}_\theta + \cos\phi\cos\theta\,\mathbf{u}_\phi). \tag{A 3}$$

$\mathbf{u}_R$, $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$ stand for unit vectors in the directions of increasing R, θ and ϕ. Hence the integral of W^2 over the infinite volume of liquid is

$$\int W^2 d\tau = B^2/3\pi r_e. \tag{A 4}$$

We next proceed to evaluate λ. We require the harmonic function whose normal derivative is given over a surface consisting of the xy plane for $R > r_e$ and the hemispherical surface bounding the electrode. This may be found by division into two parts thus:

$$\lambda = \lambda_1 + \lambda_2.$$

λ_1 is harmonic in the region $0 < R < \infty$, $0 < \theta < \frac{1}{2}\pi$ and satisfies

$$\left(\frac{\partial\lambda_1}{\partial\theta}\right)_{\theta=\frac{1}{2}\pi} = \frac{B}{2\pi R}\sin\phi.$$

λ_2 is harmonic in the region $R > r_e$, $0 < \theta < \pi$, it satisfies $\lambda_2(R, \pi-\theta, \phi) = \lambda_2(R, \theta, \phi)$ (this ensures that the normal derivative of λ_2 on $\theta = \frac{1}{2}\pi$ is zero) and is such that its normal derivative on $R = r_e$, $0 < \theta < \frac{1}{2}\pi$ is equal and opposite to that of λ_1 (this

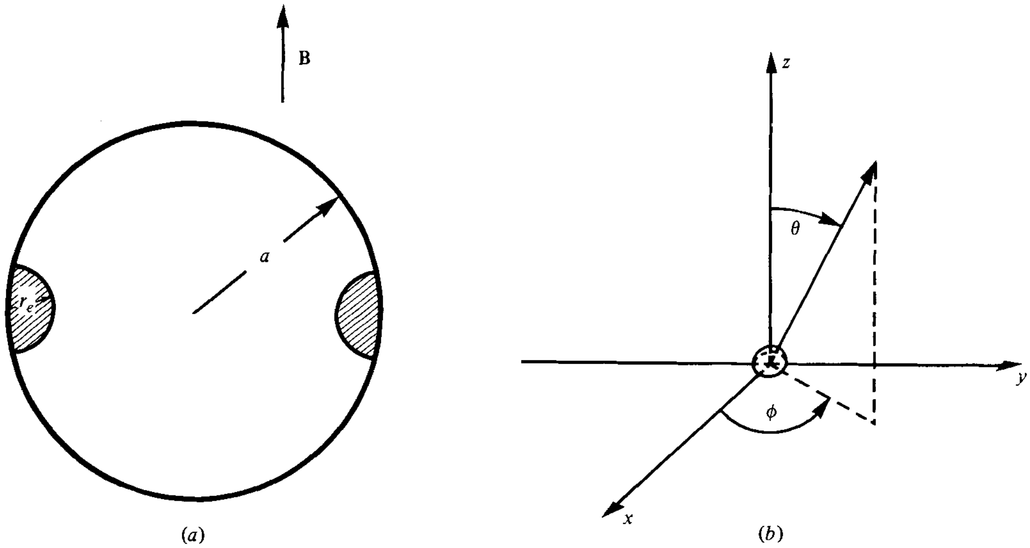

FIGURE 1. (*a*) Cross-section of flowmeter normal to pipe axis through the mid-plane of the hemispherical electrodes: a, radius of pipe; r_e, radius of electrode; **B**, direction of uniform magnetic field. (*b*) Co-ordinate system: the pipe wall is in the xy plane and the centre of the base of one hemispherical electrode coincides with the origin.

ensures that the normal derivative of λ on $R = r_e$, $0 < \theta < \frac{1}{2}\pi$ is zero, in conformity with the condition that $\partial\lambda/\partial n = W_n$, where $W_n = 0$ since $\mathbf{j}$ in $\mathbf{W} = \mathbf{B}\times\mathbf{j}$ is normal to the surface of the electrode). Straightforward integration of Laplace's equation in spherical co-ordinates yields

$$\lambda_1 = \frac{B}{2\pi R}\sin\phi\,\frac{1-\cos\theta}{\sin\theta}. \tag{A 5}$$

λ_2 may be found by using spherical harmonics and the associated Legendre functions:

$$\lambda_2 = -\frac{B}{2\pi R}\sin\phi\sum_{n=0}^{\infty}\frac{A_n}{2n+2}\left(\frac{r_e}{R}\right)^{2n+1}P^1_{2n+1}(\cos\theta), \tag{A 6}$$

where

$$A_n = \left(\frac{-1}{2}\right)^{n+1}\frac{4n+3}{(2n+2)(2n+1)^2}\,\frac{(2n+1)(2n-1)(2n-3)\dots 1}{(n+1)!}. \tag{A 7}$$

Finally

$$\lambda = \frac{B}{2\pi R}\sin\phi\left[\frac{1-\cos\theta}{\sin\theta} - \sum_{n=0}^{\infty}\frac{A_n}{2n+2}\left(\frac{r_e}{R}\right)^{2n+1}P^1_{2n+1}(\cos\theta)\right]. \tag{A 8}$$

We obtain from equations (A.3) and (A.8)

$$\int \lambda\mathbf{W}.d\mathbf{s} = \frac{B^2}{4\pi r_e}\left[1 - \sum_{n=0}^{\infty}\frac{A_n}{(2n+2)^2}P^1_{2n+1}(0)\right],$$

in which

$$P^1_{2n+1}(0) = \frac{(-1)^{n+1}}{2^n}\,\frac{(2n+1)(2n-1)(2n-3)\dots 1}{n!}.$$

A numerical evaluation of the sum gives the result

$$\int \lambda\mathbf{W}.d\mathbf{s} = 0{\cdot}201\frac{B^2}{\pi r_e}.$$

The value of ϵ is, by equations (A.1), (19), (A.4) and the above result:

$$\epsilon = 0{\cdot}14(a/r_e)^{\frac{1}{2}}$$

(where the integrals have been doubled).

It may be shown, by taking the integrals of W^2 and λW_n between the limits r_e and a instead of r_e and ∞, that the contribution to their difference from regions beyond $R = a$ is approximately a fraction $r_e/2a$ of the total and therefore small. Consequently the initial assumption is verified.

We note also that both $\mathbf{W}$ and $\nabla\lambda$ tend to 0 as $R \to \infty$, in conformity with the boundary condition expressed by equation (8), §2.2.

Appendix B. Cases where ϵ is theoretically infinite

In addition to the point-electrode flowmeter with magnetic field locally uniform at an electrode, there are three other cases we have noticed where $\epsilon = \infty$. The second and third of these are both for the point-electrode flowmeter, either when the magnetic field at points in the locality of the electrodes is non-uniform, finite and non-zero or when it is zero at the electrodes but on approaching them does not tend to zero rapidly enough to ensure the convergence of I (see §4). These three cases are due essentially to the infinite value of the virtual current at a point electrode, since $\mathbf{W} = \mathbf{B} \times \mathbf{j}$. The fourth case is rather different and reflects the symmetry of $\mathbf{W}$ in $\mathbf{B}$ and $\mathbf{j}$. It occurs when there is a discontinuity in magnetic potential across a line on the inside surface of the duct of a flowmeter with any kind of electrodes. (Such a discontinuity is approximated in practice in, for example, a magnet consisting of a thin current-carrying wire laid on the inside surface of an insulated iron duct, or in a magnet consisting of a current sheet laid on the inside surface of an iron duct when the sheet is partially covered by a thin layer of permalloy, the discontinuity occurring at the boundary of the permalloy.) Near a small segment S of the line the virtual current is uniform and the magnetic field is of order r^{-1}, where r is the perpendicular distance of a point from the line. $|\nabla\lambda|$ does not diverge as $r \to 0$. Hence $|\mathbf{W} - \nabla\lambda| \sim r^{-1}$ and the part of the integral in the numerator of equation (19) conducted over a small half-cylinder in the channel whose axis coincides with the segment S diverges logarithmically.

Appendix C. Proof that the flow given by equation (14) corresponds to a maximum of flowmeter response

Consider a flow differing from $\mathbf{v}$ (as given by equation (14)) and let it be $\mathbf{v} + \mathbf{v}'$. Let it be constrained in the same way as $\mathbf{v}$, so that $\mathbf{v}'$ satisfies equations (7), (8), (10) and the relationship

$$\int (\mathbf{v} + \mathbf{v}')^2 \, d\tau = K. \tag{C 1}$$

Equation (3.1) together with equation (9) (for $\mathbf{v}$) gives

$$\int \mathbf{v} . \mathbf{v}' \, d\tau < 0. \tag{C 2}$$

The response due to $\mathbf{v}$ is given by equation (2). This, with equations (14), (7) and (10), gives

$$U = \int \mathbf{v} . (\mathbf{W} - \nabla\lambda)\, d\tau = -\frac{1}{2\gamma}\int (\mathbf{W} - \nabla\lambda)^2\, d\tau.$$

This is greater than zero if we take the negative value of γ. The response due to $\mathbf{v} + \mathbf{v}'$ is $U + U'$. By equations (2), (7), (8) and (10) (for $\mathbf{v}'$) we have

$$U' = \int \mathbf{v}' . \mathbf{W}\, d\tau = \int \mathbf{v}' . (\mathbf{W} - \nabla\lambda)\, d\tau.$$

We now use equation (14) to obtain

$$U' = -2\gamma \int \mathbf{v}' . \mathbf{v}\, d\tau.$$

Now by equation (C.2) U' is negative (for the negative value of γ). Hence $U + U' < U$ for all $\mathbf{v}'$ so the flow given by equation (14) corresponds to a maximum of the absolute value of the response.

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